# MATH 20002: COMBINATORICS

Solution: Sample Exam Question

**Solution of Problem 1**:

(a) [Bookwork] A planar graph is a graph $G$ which can be drawn in the plane in such a way that no two of its edges intersect (except possibly at a vertex).

(b) [Bookwork] Euler's formula states that given any connected planar graph $G = (V, E)$ together with a planar drawing with face set $F$, we have

$$|V| - |E| + |F| = 2.$$

Since the boundary of every face contains at least 3 edges and every edge is on the boundary of at most 2 faces, we have by double-counting that $3|F| \leq 2|E|$, so that

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|,$$

or

$$|E| \leq 3|V| - 6. \quad (*)$$

By the Handshaking Lemma,

$$2|E| = \sum_{x \in V} \deg_G(x),$$

so if every vertex in the connected planar graph $G$ had degree at least 6, we would have $2|E| \geq |V| \cdot 6$, which combined with $(*)$ would yield

$$3|V| \leq |E| \leq 3|V| - 6,$$

a contradiction. Therefore $G$ contains at least one vertex of degree at most 5.

(c) [Bookwork] The chromatic number $\chi(G)$ of a graph $G$ is defined to be the least integer $k$ such that the vertices of $G$ can be coloured with $k$ colours in such a way that no two adjacent vertices receive the same colour.

(d) [Problem Sheet] We may without loss of generality assume that the planar graph $G$ is connected, since otherwise we may colour each connected component separately. We prove that $\chi(G) \leq 6$ for any planar connected graph $G$ by induction on the number of vertices of $G$.

The statement is clearly true for any planar connected graph on at most 6 vertices. Suppose then that $G$ is a planar connected graph on $n > 6$ vertices, and assume that the statement holds for all planar connected graphs on strictly fewer vertices.

By (b), every planar connected graph $G$ has a vertex of degree at most 5. Label this vertex $v$, and remove it (and all incident edges) from $G$ to obtain a graph $G'$. This graph $G'$ has $n-1$ vertices and is clearly planar. By the inductive hypothesis, every connected component of $G'$ can be coloured with at most 6 colours, so $G'$ can be coloured with at most 6 colours. But the vertex $v$ has at most 5 neighbours, which in any (valid) colouring of $G'$ receive at most 5 colours, so there is (at least) one colour left for $v$. In this way we obtain a valid colouring of $G$, which completes the induction.

[An alternative way of dealing with the connectedness issue is to observe that (b) is true even when the graph is not necessarily connected (again, by considering a connected component).]

(e) [Unseen]

1. We proved in (d) that any planar graph can be coloured with at most 6 colours. This means that when $G$ is a planar graph, the chromatic polynomial $P_G(\lambda)$ must take a non-zero value for all integer values of $\lambda \geq 6$. This means that all integer roots of $P_G(\lambda)$ must be less than or equal to 5.
   [In fact, $\chi(G) = \min\{\lambda \in \mathbb{N} : P_G(\lambda) > 0\}$.]
2. Let $T$ be a tree on $n$ vertices. If $n = 1$, it is easy to verify that $P_T(\lambda) = \lambda$ holds (since the one vertex can be colour with any one of the $\lambda$ colours). Suppose then that $n \geq 2$ and that the formula holds for all trees on fewer vertices. Since $T$ is a tree on at least 2 vertices, it has a leaf $x$, say. Remove this leaf (and its incident edge $xy$, say) from $T$ to obtain a tree $T'$ on $n-1$ vertices. By the inductive hypothesis, $P_{T'}(\lambda) = \lambda(\lambda-1)^{n-2}$. Observe that any valid colouring of $T'$ with at most $\lambda$ colours gives rise to precisely $\lambda - 1$ colourings of $T$, since we may colour $x$ with any of the $\lambda - 1$ colours that haven't been used on $y$. Hence $P_T(\lambda) = (\lambda - 1) \cdot P_{T'}(\lambda) = \lambda(\lambda-1)^{n-1}$, as claimed. This completes the inductive step of the proof.
3. [I would strongly recommend drawing a 4-cycle when tackling this question. You may also want to experiment with small values of $\lambda$ before attempting to formulate a more general argument.]
   Label the vertices of the 4-cycle $x, y, z, w$, and let its edges be $xy, yz, zw, xw$. We have $\lambda$ colours at our disposal. Clearly $x$ and $w$ can be coloured in $\lambda(\lambda-1)$ ways (since they are adjacent they must receive different colours). There are now two (distinct) possibilities for $y$: either we re-use the colour we used for $w$,

or we don't. Let's first deal with the case where we choose the same colour for $y$ as we did for $w$. In that case we have precisely $\lambda - 1$ colours left to colour $z$ with. The number of colourings of $C_4$ in which $y$ and $w$ receive the same colour is therefore $\lambda(\lambda-1)^2$. If we insist that $y$ receive a different colour from $w$, then we have $\lambda - 2$ choices for the colour of $y$, and subsequently $\lambda - 2$ choices for the colour of $z$. The number of colourings of $C_4$ in which $y$ and $w$ receive different colours is therefore $\lambda(\lambda-1)(\lambda-2)^2$. By the addition rule of counting, the total number of colourings of the 4-cycle with at most $\lambda$ colours is

$$P_{C_4}(\lambda) = \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)^2 = (\lambda-1)^4 + (\lambda-1).$$

**Notes:**

- Not all exam questions will have the exact same ratio of seen to unseen material.
- Some parts of some questions are designed to be difficult. Do those (parts of) questions you know how to do first.
- Any facts (lemmas/propositions/theorems) from the lecture notes which you use must be stated clearly. For example, you must give the meaning of each letter used in Euler's formula. If you are using an earlier part of a question to justify your argument, please refer to it concisely.
- Remember, the overall clarity of your argument counts towards your mark.

*Any comments or corrections should be sent to Dr Julia Wolf at julia.wolf@bristol.ac.uk.*