MATH 20002: COMBINATORICS

Sample Exam Question

Problem 1. [25 marks]

(a) (2 marks) Describe informally what we mean by a planar graph.

(b) (6 marks) Show that any connected planar graph G has a vertex of degree at most 5.

(You may assume Euler's formula, but all other facts about planar graphs need to be derived.)

(c) (2 marks) Define what we mean by the chromatic number $\chi(G)$ of a graph G.

(d) (5 marks) Show that for any planar graph G, $\chi(G) \leq 6$.

(e) Given an integer $\lambda \geq 0$ and a graph G, we denote the number of (valid) colourings of its vertex set with at most λ colours by $P_G(\lambda)$. (Colourings are counted as distinct even if they differ only by a permutation of the colours.) It turns out that $P_G(\lambda)$ is a polynomial in λ with integer coefficients, known as the *chromatic polynomial* of the graph G. For example, the chromatic polynomial of the complete graph K_n on n vertices is $P_{K_n}(\lambda) = \lambda(\lambda-1)\cdots(\lambda-(n-1))$.

1. (2 marks) Let G be a planar graph. Explain why all integer roots of the chromatic polynomial $P_G(\lambda)$ must be less than or equal to 5.
2. (4 marks) Show by induction on the size of the vertex set that the chromatic polynomial of a tree T on n vertices is
$$P_T(\lambda) = \lambda(\lambda-1)^{n-1}.$$
3. (4 marks) Derive a formula for the chromatic polynomial $P_{C_4}(\lambda)$ of the 4-cycle.