MATH 20002: COMBINATORICS

Solutions Problem Sheet 8:
Planar Graphs and Graph Colouring

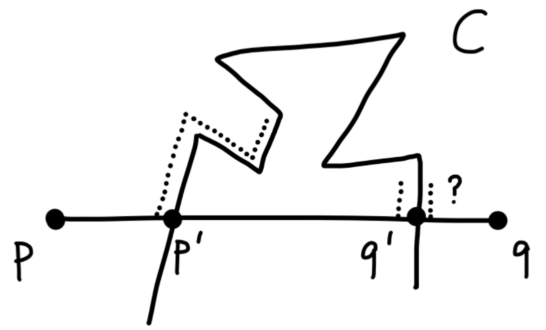
Solution of Problem 1:

(a) The individual steps can be justified as follows (most of the work lies in the correct set-up, given in the question).

1. The parity of a point p moving along a line segment not intersecting C cannot change since a ray through p in the fixed direction always passes through C the same number of times. (The only potentially awkward situation arises when the ray through p passes through a corner of C, but even in this case the parity does not change because of the convention that an intersection only be counted if the two line segments of C meeting at that corner are on opposite sides of the ray.

   It is immediate that if any point $p \in A$ is joined to any point $q \in B$ by a polygonal arc, then this arc must intersect C for otherwise the parity of all points on the arc, and in particular of p and q, would be the same.

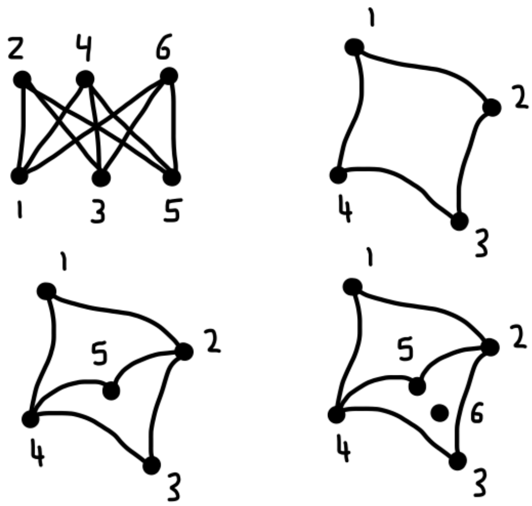
2. Without loss of generality assume that p and q are both in A. If the straight line segment ℓ connecting p and q does not intersect C, then it is the desired arc. Otherwise, let p' be the first point of intersection of the segment ℓ with C, and let q' be the last such point. Construct a polygonal arc as follows: follow ℓ from p to just before p', then turn off and follow C until C returns to ℓ at q' (dotted line below).



   We need to show that this (polygonal) arc intersects ℓ between q' and q, rather than between p' and q', so that we can continue along ℓ all the way to q without intersecting C. But note that if we were to cross the point q' along ℓ, the parity would change, contradicting the fact that both p and q have the same parity, and by 1), so do all points on the polygonal arc we have constructed until we meet ℓ again near q'.

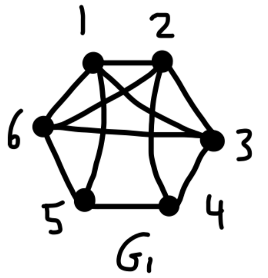
3. By definition, the set A and B partition $\mathbb{R}^2 \setminus C$, and thus the polygonal Jordan curve theorem is proved: any polygonal curve C divides the plane into precisely two connected parts, the interior and the exterior, and C is the boundary of both regions.

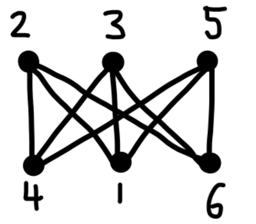
(b) Suppose there was a planar drawing of $K_{3,3}$. In any such drawing, the cycle $1, 2, 3, 4$ divides the plane into two disjoint regions (by the Jordan curve theorem). The path $2, 5, 4$ must lie entirely within one of these regions, either the interior or the exterior. Without loss of generality assume the path lies in the interior, as illustrated on the bottom left (the other case is identical). Again, this path divides the interior into two disjoint regions, and vertex 6 must lie in one of these regions or in the exterior of the cycle $1, 2, 3, 4$. Assume vertex 6 is located as illustrated on the bottom right (the other cases are identical). But any arc connecting 6 to 1 must cross the Jordan curve formed by $2, 3, 4, 5$, since 6 lies in its interior and 1 in its exterior, so in particular the arc connecting 6 to 1 must cross one of the arcs 23, 34, 45 or 25, contradicting the assumption that we had a planar drawing to start with.



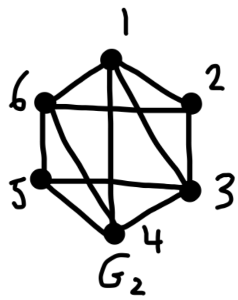
Solution of Problem 2:

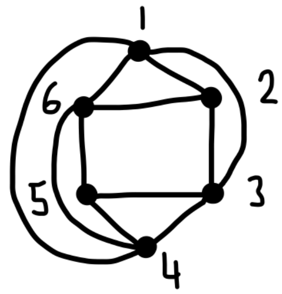
G_1 contains a $K_{3,3}$ as a subgraph (see right) and is therefore, by Kuratowski's theorem, non-planar.



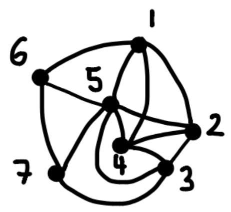


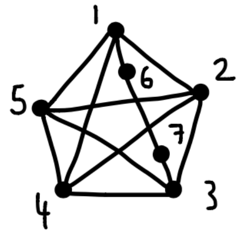
G_2 is easily seen to be planar. It suffices to exhibit a planar drawing (right).





G_3 is not planar since it contains a subdivision of K_5 as a subgraph, shown on the right. Again, we appeal to Kuratowski's theorem (Theorem 8.8).





Solution of Problem 3:

(a) If $n = 1$ or 2, then the statement is vacuously true. Let $n \geq 3$, and suppose for the purposes of obtaining a contradiction that every vertex has degree at least 6. It follows from the handshaking lemma that

$$2|E| = \sum_{x \in V} \deg_G(x) \geq 6n,$$

but if G is planar and connected on at least three vertices, then we also know from Theorem 8.11 that it can have at most $3n - 6$ edges. It follows that

$$6n - 12 = 2(3n - 6) \geq 6n,$$

which is a contradiction. It follows that there must be a vertex of degree < 6 in G.

(b) Any connected planar graph in which every vertex has degree at least 5 must satisfy

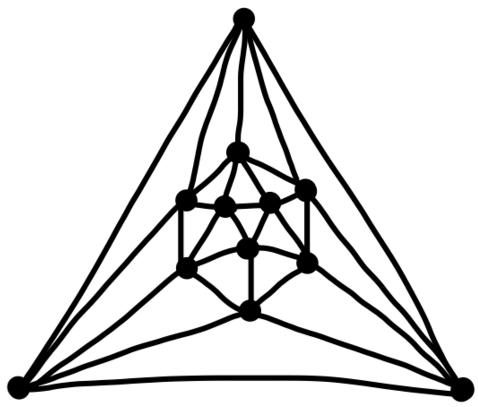
$$2|E| = \sum_{x \in V} \deg_G(x) \geq 5n,$$

by the Handshaking lemma. Moreover, by Theorem 8.11 we have that $|E| \leq 3n-6$, and hence

$$6n - 12 = 2(3n-6) \geq 5n,$$

or $n \geq 12$.

(c) The graph of an icosahedron is an example. (Imagine the icosahedron drawn on the surface of a balloon. Pierce it at one point (not coinciding with any vertices or edges) and stretch out the surface of the balloon in the plane. We'll come back to this after Easter.)



Solution of Problem 4:

(a) Let $G = (V, E)$ be a planar graph and let $G' = (V, E')$ its complement. Set $n := |V|$, so by definition $|E'| = \binom{n}{2} - |E|$. From Theorem 8.11 we know that since G is planar,

$$|E| \leq 3n - 6.$$

If in addition G' were planar, then we would have to have

$$|E'| \leq 3n - 6,$$

but

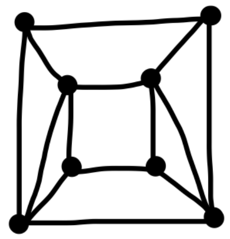
$$\frac{n(n-1)}{2} - (3n-6) \leq \binom{n}{2} - |E| = |E'| \leq 3n - 6$$

is equivalent to

$$n(n-1) \leq 4(3n-6),$$

which has no solution for $n \geq 11$. Thus the complement of any planar graph on ≥ 11 vertices must be non-planar.

(b) You can check that the following graph on 8 vertices is both planar and isomorphic to its own complement.



Solution of Problem 5:

(a) As in the proof of Theorem 8.11, we have that N, the number of pairs (e, f) where $e \in E$ is an edge and $f \in F$ is a face of the drawing, and e forms part of the boundary of f, is at most $2|E|$ but also at least $4|F|$, since the boundary of a face in a drawing of a triangle-free graph must consist of at least 4 edges. It follows that

$$|E| = |V| + |F| - 2 \leq |V| + \frac{1}{2}|E| - 2,$$

and thus

$$|E| \leq 2(|V| - 2) = 2n - 4.$$

(b) To see that this bound is best possible in general, consider the graph on vertex set $u, v, x_1, x_2, \ldots, x_{n-2}$, which has edges ux_i and vx_i for $i \in [n-2]$. This graph is clearly planar and contains no triangles. It has $2(n-2) = 2n-4$ edges.

(c) Since $K_{3,3}$ is triangle-free, if it were planar we would have, by (a), that $|E| \leq 2|V| - 4$. But $|V| = 6$ and $|E| = 9$, which yields $9 \leq 2 \cdot 6 - 4 = 8$, a contradiction. It follows that $K_{3,3}$ is not planar.

Solution of Problem 6:

(a) Again, as in the proof of Theorem 8.11, we have that N, the number of pairs (e, f) where $e \in E$ is an edge and $f \in F$ is a face of the drawing, and e forms part of the boundary of f, is at most $2|E|$ but also at least $3|F|$, since the boundary of each face has at least 3 edges. It follows that

$$\frac{3}{2}|F| \leq |E| = |V| + |F| - 2,$$

and thus

$$|F| \leq 2(|V| - 2) = 2n - 4.$$

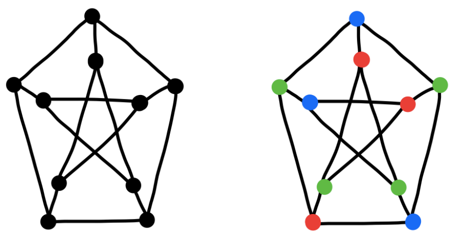
(b) As in Problem 5 above, if G is triangle-free then we have $2|E| \geq 4|F|$, and thus

$$2|F| \leq |E| = |V| + |F| - 2,$$

or $|F| \leq |V| - 2 = n - 2$ as claimed.

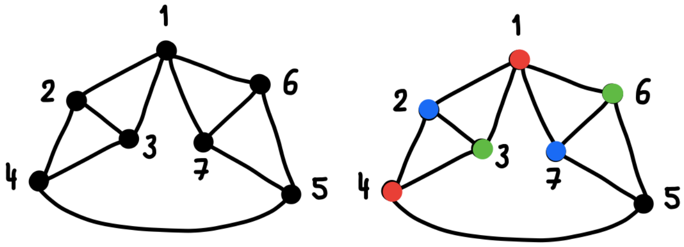
Solution of Problem 7:

(a) The Petersen graph contains odd-length cycles, and therefore its chromatic number must be at least 3. It is possible to colour the Petersen graph using 3 colours, for example as follows.



(b) The graph G has chromatic number at least 3 since it contains a triangle. Suppose then that we could colour G with just 3 colours. Vertices 1, 2 and 3 must receive distinct colours. In order to avoid introducing a 4th colour, vertex 4 must receive the same colour as vertex 1. Also, vertices 6 and 7 must receive a colour distinct from 1, hence if we want to stick with 3 colours then they must receive the same two colours as vertices 2 and 3. As before, vertex 5 needs to receive the same colour as vertex 1. But this is impossible since vertices 4 and 5 are connected by an edge. It is therefore impossible to colour G with 3 colours only.
However, a colouring with 4 colours is easy to find (on the right below, we could,

for example, simply give vertex 5 the colour purple). We conclude that $\chi(G) = 4$.



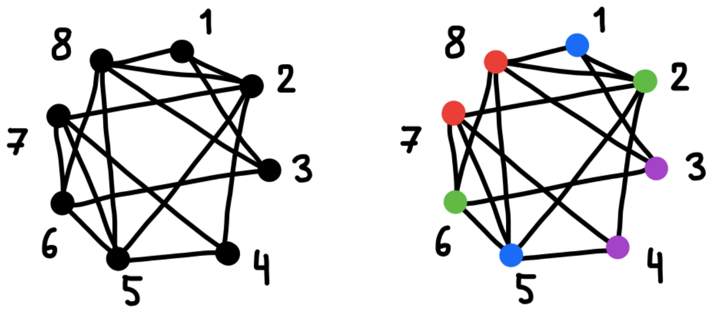
Solution of Problem 8:

(a) If $n \leq k+1$, then $G_{n,k} = K_n$ because any two vertices are within distance k of each other. Hence $\chi(G_{n,k}) = \chi(K_n) = n$. If $n > k+1$, then $G_{n,k}$ contains K_{k+1} because any two vertices in $\{1, 2, \ldots, k+1\}$ are within distance k of each other. Thus $\chi(G_{n,k}) \geq \chi(K_{k+1}) = k+1$. On the other hand, there is a valid colouring of $G_{n,k}$ with $k+1$ colours. For example, we may colour the vertices with periodically, giving $j \in \{1, 2, \ldots, n\}$ the colour $s \in \{1, 2, \ldots, k+1\}$ if $j \equiv s \mod (k+1)$. Then any two vertices of the same colour are at distance $> k$, hence they cannot be adjacent. It follows that $\chi(G_{n,k}) \leq k+1$, and therefore that $\chi(G_{n,k}) = k+1$ whenever $n > k+1$.

(b) Note that $G_{n,k}$ is connected because vertices $1, 2, \ldots, n$ always form a path. By Euler's theorem (Corollary 4.17), $G_{n,k}$ is Eulerian if and only if every vertex of $G_{n,k}$ has even degree.
As in (a), if $n \leq k+1$, then any two vertices of $G_{n,k}$ are adjacent, so every vertex has degree $n-1$. Thus for $n \leq k+1$, $G_{n,k}$ is Eulerian if and only if n is odd.
If $n > k+1$, then the degree of vertex 1 is k, while the degree of vertex 2 is $k+1$. Since one of k and $k+1$ must always be odd, $G_{n,k}$ is never Eulerian for $n > k+1$.

Solution of Problem 9:

(a) The graph consists of one vertex per workshop. Two vertices are connected by an edge if the corresponding workshops have an instructor in common. The problem of determining the minimum number of time slots required to run these workshops corresponds to finding the chromatic number of this graph (shown on the left below): given a (valid) colouring of the vertices of this graph, two workshops that receive the same colour can be held at the same time as there is no instructor conflict.



(b) In order to determine the chromatic number of the graph, we observe that the vertices 2, 4, 5 and 7 form a K_4. Therefore the chromatic number of the graph is at least 4. On the right we have exhibited an example of a (valid) colouring with 4 colours, so the chromatic number of the graph equals 4.
We conclude that Brizzle Yoga requires 4 separate time slots to run the 8 workshops with the instructors as indicated.

Solution of Problem 10:

(a) To see that $\chi(G) \geq \omega(G)$, note that if a graph G contains a clique of size k, then at least k colours will be needed to colour its vertices. For the second part of the question, note that in a graph of chromatic number $\chi(G)$, the vertices can be partitioned into $\chi(G)$ classes each of which is an independent set of size at most $\alpha(G)$. Therefore $|V| \leq \alpha(G) \cdot \chi(G)$, or $\chi(G) \geq |V|/\alpha(G)$ as claimed.

(b) Suppose that we are given a colouring C of G using $k := \chi(G)$ colours. Observe that if $v \in V$ is coloured red, say, in the colouring C, then none of its neighbours can have received the colour red. So there are at most $n - d$ red vertices in this colouring of G. The same is of course true for any of the other colours used by C. Summing over all k colours, we find that

$$n = \sum_{j=1}^{k} |\{v \in V : \ v \text{ is coloured with colour } j\}| \leq (n-d) \cdot k,$$

from which it follows that

$$k \geq \frac{n}{n-d}.$$

Solution of Problem 11:

(a) Euler's formula (Theorem 8.10) tells us that any connected planar graph $G = (V, E)$ whose given planar drawing has face set F satisfies

$$|V| - |E| + |F| = 2.$$

As in the proof of Theorem 8.11, we count the number N of pairs (e, f) where $e \in E$ is an edge and $f \in F$ is a face of the drawing, and e forms part of the boundary of f. Since every edge borders on at most two faces, we have $N \leq 2|E|$. Furthermore, if G is triangle-free, then any face must be formed by at least 4 edges, so $N \geq 4|F|$. It follows that $4|F| \leq 2|E|$, and therefore

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{1}{2}|E| = |V| - \frac{1}{2}|E|.$$

It follows that for a connected triangle-free planar G, $|E| \leq 2|V| - 4$ (note that we already proved this fact in Problem 5 above).

Now by the handshaking lemma, if every vertex in G had degree at least 4, we would have

$$4|V| \leq \sum_{x \in V} \deg_G(x) = 2|E| \leq 2(2|V| - 4) = 4|V| - 8,$$

a clear contradiction. Therefore G must have at least one vertex of degree at most 3.

(b) We show that any triangle-free planar graph G is 4-colourable by induction on $n := |V|$. The cases $n \leq 4$ are all trivial. Suppose then that $n > 4$ and that the result holds for all triangle-free planar graphs on strictly fewer vertices.

Consider a largest connected component G' of G, which is itself planar and triangle-free. By (a), G' has a vertex of degree at most 3, call it v. Remove v from G' to obtain a graph G'' on $< n$ vertices, which is clearly still triangle-free and planar (but may no longer be connected). By the inductive hypothesis, G'' can be coloured with at most 4 colours. Since v was connected to at most 3 vertices in G'', we may conclude that we can also colour G' with at most 4 colours (since there is at least one colour which may be used for v as it was not used by its neighbours). Any other connected components (if there are any) may also be coloured with at most 4 colours by the inductive hypothesis, and these colourings do not interfere with each other since there are no edges between distinct connected components. We have thus obtained a (valid) colouring of G using 4 colours only.

Solution of Problem 12:

(a) Let G be a planar bipartite graph, and let v^* be a vertex in the dual graph G^*. Then $\deg_{G^*}(v^*)$ is the number of edges bounding the face in the planar drawing of G that corresponds to v. The edges bounding the face form a cycle in G. Since G is bipartite, the number of edges in any cycle is even. Thus $\deg_{G^*}(v^*)$ is even for all v^* in G^*.

(b) We proceed exactly as in the proof of Theorem 9.6, by induction on $n := |V|$. When $n \leq 6$, the result is trivially true. Suppose therefore that $n > 6$ and that the result holds for all planar graphs on strictly fewer vertices. By Problem 3(a) on Sheet 8 we know that any planar graph has a vertex of degree at most 5. Call this vertex $v \in V$. Let $G' := G \setminus \{v\}$. The graph G' is a planar graph on $n - 1$ vertices, so by the inductive hypothesis G' can be coloured with at most 6 colours. But $N_G(v)$ consists of at most 5 vertices, which use at most 5 colours, so we must have at least one admissible colour left to colour v with.

Any comments or corrections should be sent to Dr Julia Wolf at julia.wolf@bristol.ac.uk.