MATH 20002: COMBINATORICS

Problem Sheet 8:
Planar Graphs and Graph Colouring

Problem 1. The goal of the first part of this exercise is to rigorously prove the Jordan curve theorem for polygonal arcs, without relying on any geometric intuition.

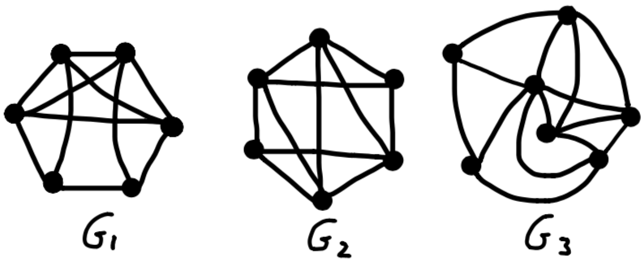
(a) A *polygonal arc* is an arc that consists of finitely many straight line segments. A *polygonal curve* is polygonal arc whose end points coincide.

Let C be such a polygonal curve in $\mathbb{R}^2$. Choose a fixed direction in the plane that isn't parallel to any of the sides of C. (Why is this always possible?) Define two classes A and B as follows: a point $p \in \mathbb{R}^2 \setminus C$ belongs to A if the ray through p in the fixed direction intersects C an even number of times; a point $p \in \mathbb{R}^2 \setminus C$ belongs to B if the ray through p in the fixed direction intersects C an odd number of times. (If a ray intersects C at a "corner", we shall only count the intersection if the two line segments of C meeting at that corner are on opposite sides of the ray.) We say two points $p, q \in \mathbb{R}^2 \setminus C$ have the same *parity* if they belong to the same class, A or B.

1. Show that all points of $\mathbb{R}^2 \setminus C$ on any line segment not intersecting C have the same parity, and deduce that if any point $p \in A$ is joined to any point $q \in B$ by a polygonal arc, then this arc must intersect C.
2. Show that any two points of the same class, A or B, can be joined by a polygonal arc which does not intersect C.
3. Conclude the proof of the theorem by identifying A with the exterior of C, and B with the interior of C.

(b) Use the (full) Jordan curve theorem to show that the graph $K_{3,3}$ is not planar.

Problem 2. For each of the following graphs determine whether or not the graph is planar. Justify your answers.



Problem 3.

(a) Prove that any connected planar graph must have a vertex of degree at most 5.
(b) Prove that a connected planar graph in which each vertex has degree at least 5 must have at least 12 vertices.
(c) Find a connected planar graph all of whose vertices have degree 5.

Problem 4. The *complement* of a graph $G = (V, E)$ is defined to be the graph $G' = (V, E')$ where $E' = \{xy : x, y \in V\} \setminus E$. For example, the complement of a C_4 is a graph on the same vertex set consisting of two independent edges (the "diagonals").

(a) Prove that the complement of a planar graph on at least 11 vertices is non-planar.
(b) Find a simple planar graph on 8 vertices whose complement is planar.

Problem 5. Let $G = (V, E)$ be a connected planar graph, and let $n := |V|$.

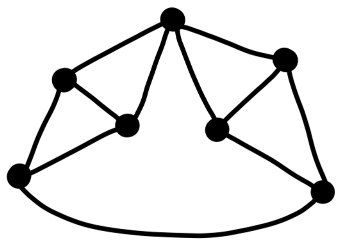
(a) Show that if $n \geq 3$ and G is triangle-free, then $|E| \leq 2n - 4$.
Hint: You may wish to follow the approach in Theorem 8.11.
(b) Show that the bound in (a) is best possible in general.
(c) Conclude that $K_{3,3}$ is not planar.

Problem 6. Let $G = (V, E)$ be a planar graph, and let $n := |V|$.

(a) Show that if $n \geq 3$, then any planar drawing of G has at most $2n - 4$ faces.
(b) Show that if G is triangle-free, then any planar drawing of G has at most $n - 2$ faces.

Problem 7.

(a) Show that the Petersen graph has chromatic number 3.
(b) Let G be the graph below. Explain why $\chi(G) = 4$.



Problem 8. Let $n, k \in \mathbb{N}$, and let $G_{n,k}$ be the graph with vertex set $[n] := \{1, 2, \ldots, n\}$ and edge set $\{ij : i, j \in [n], |i - j| \leq k\}$.

(a) For all values of $n, k \in \mathbb{N}$, determine the chromatic number $\chi(G_{n,k})$.

(b) For which values of n, k is $G_{n,k}$ Eulerian?

Problem 9. Brizzle Yoga is planning to run a series of 8 workshops with plenty of hands-on instruction, so they have assigned two or three yoga teachers to each of the workshops as follows:

W1: Charley, Fiona, Naomi | W2: Charley, Xander, Gerry | W3: Fiona, Bella | W4: Mel, Xander, Kylie | W5: Mel, Tom, Gerry | W6: Tom, Bella | W7: Tom, Xander | W8: Fiona, Bella, Gerry

Two workshops cannot be held in the same 60-minute time slot if some staff member is assigned to both. The problem is to determine the minimum number of time slots required to run these workshops.

(a) Recast this problem as a question about colouring the vertices of a particular graph. Draw the graph and explain what the vertices, edges, and colours mean.

(b) How many colours are needed to colour the vertices of this graph? What timetable does this imply for the workshops?

Problem 10.

(a) Let $G = (V, E)$ be a graph. Denote by $\omega(G)$ the size of a largest clique in G, and by $\alpha(G)$ the size of a largest independent set in G. Show that $\chi(G) \geq \omega(G)$ and $\chi(G) \geq |V|/\alpha(G)$.

(b) A graph is said to be *d-regular* if every vertex has degree exactly d. Let G be a d–regular graph on n vertices. Prove that

$$\chi(G) \geq \frac{n}{n-d}.$$

Problem 11.

(a) Show that any connected triangle-free planar graph has at least one vertex of degree at most 3.

(b) Show that any triangle-free planar graph is 4-colourable.

Problem 12.

(a) Suppose that G is a planar bipartite graph. Show that all vertices of the dual graph G^* are of even degree.

(b) By mimicking Case 1 in the proof of the 5-colour theorem (Theorem 9.6), give a quick argument to show that any planar graph is 6-colourable.

Any comments or corrections should be sent to Dr Julia Wolf at julia.wolf@bristol.ac.uk.