MATH 20002: COMBINATORICS

Solutions Problem Sheet 5: Introduction to Graph Theory and Bipartite Graphs

Solution of Problem 1:

(a) Suppose the graph has n vertices. Since no vertex in the graph can have degree strictly greater than $n-1$, if the degrees of all n vertices are distinct then they must take values $0, 1, 2, \ldots, n-1$. But the vertex of degree $n-1$ must be connected to all other vertices in the graph, so there cannot be a vertex with degree 0.

(b) This is an easy corollary of a statement about spanning trees which we will prove in a couple of weeks' time, namely that every connected graph has a spanning tree. Given such a spanning tree, we can remove a vertex v from G which is a leaf of the spanning tree. The remaining vertices of the graph are still connected (via what remains of the spanning tree after removal of v). For a proof which does not assume this result, let $G = (V, E)$ be a connected graph and let P be a longest path in G. We claim that we can remove one of the end vertices of P. Suppose v is such an end vertex. Since P is assumed to be a longest path in G, v can only have neighbours amongst the vertices of P. Removing v therefore cannot create a new connected component.

Solution of Problem 2: We only need to concern ourselves with the case of 2 vertices having odd degree (the case of 0 vertices is covered by Corollary 4.17 in the notes, and by the Handshaking lemma it is impossible for a graph to have only one vertex of odd degree). If there is a Eulerian trail (whose end points do not join up–we have already shown that if there is a Eulerian circuit then every vertex must have even degree, so this case can be excluded), then there there are precisely two vertices of odd degree, namely the start- and end-vertex of the trail. We thus focus on the reverse implication, namely that if G is a connected graph with precisely two vertices of odd degree, then there must be a Eulerian trail.

Suppose the two vertices of odd degree are labeled x and y. Add an artificial edge xy to the graph G to obtain a (connected) graph G' all of whose degrees are even. By Corollary 4.17, it must have a Eulerian circuit. In order to obtain the Eulerian trail for G, simply delete the edge xy from the Eulerian circuit for G'.

Solution of Problem 3: We need to show that it is impossible to get stuck, namely that all of the edges are used in Fleury's algorithm. Where could we possibly get stuck? Since all degrees in the graph are even, we can always move out of a vertex (using a bridge if necessary) unless we are back at the vertex where we started, call it x_1. So suppose we are stuck at x_1. The remaining edges of the graph at this time (which have not been traversed) form (an unknown number of) connected components. But in each component, every vertex has even degree so each component is Eulerian. That is, the final graph G_m is a union of several connected components, each of which has a Eulerian circuit.

Now consider the last time the path chosen by Fleury's algorithm used a vertex from one of the connected components, say the vertex x_ℓ for some $\ell < m$. (There must be such a vertex otherwise the original graph G would not be connected.) Then removing the edge $x_\ell x_{\ell+1}$ disconnects the graph G_ℓ. But at that time the algorithm had at least two other choice of edges from x_ℓ, in the Eulerian component containing x_ℓ, neither of which is a bridge. So Fleury's algorithm would not have chosen the edge $x_\ell x_{\ell+1}$ in the first place.

Solution of Problem 4: If you follow through the proof of Dirac's theorem (Theorem 4.20), you'll find that we used the condition on the minimum degree of the graph, $\delta(n) \geq n/2$, in two places: first, to argue that the graph $G = (V, E)$ is connected. But for this the condition $\deg_G(x)+\deg_G(y) \geq n$ suffices: if $xy \notin E$, then $N(x)$ and $N(y) \subseteq V\setminus\{x, y\}$ which has size $n-2$, but $|N(x)|+|N(y)| \geq n$, so $N(x)\cap N(y) \neq \emptyset$. Second, in the final paragraph we had $N(x_1) \subseteq \{x_2, \ldots, x_{\ell-1}\}$ and $N(x_\ell) \subseteq \{x_2, \ldots, x_{\ell-1}\}$, and in particular that $N(x_1)$ and $N(x_\ell)^+ := \{x_i : x_{i-1} \in N(x_\ell)\}$ are disjoint subsets of $\{x_2, \ldots, x_\ell\}$. But the latter is a set of size $\ell - 1 < n$, while again $|N(x_1)| + |N^+(x_\ell)| \geq n$ and hence $N(x_1) \cap N^+(x_\ell) \neq \emptyset$. But this contradicts the observation, made earlier on in the proof, that the edges $x_1 x_i$ and $x_{i-1}x_\ell \in E$ could not both be present for any $i = 2, \ldots, \ell - 1$.

Solution of Problem 5: As indicated in the question, we shall prove this by induction on $n := |V|$. When $n = 2$ there is nothing to prove. Suppose then that we have graph $G = (V, E)$ of order $n \geq k + 1$ which does not contain P_k, a path of length k. Without loss of generality G is connected since if G has connected components $G_1, \ldots, G_r$ of orders $n_1, \ldots, n_r$, then by the inductive hypothesis the number of edges in each G_i is at most $(k - 1)n_i/2$, so that $|E| \leq \sum_{i=1}^{r}(k - 1)n_i/2 = (k - 1)n/2$. Also without loss of generality we may assume that $k < n$ since otherwise we would be aiming for the bound $|E| \leq (n - 1)n/2 = \binom{n}{2}$, which is trivial.

To conclude the proof we shall show that G must contain a vertex of degree $< k/2$. Note that if we have such a vertex $v \in V$, then removing v from the graph G leaves us with a graph G' of order $n-1$ which contains no P_k and therefore, by the induction hypothesis, has at most $(k-1)(n-1)/2$ edges. Adding v back in yields that G has at most $(k-1)n/2$ edges.

The proof that G contains a vertex of degree $< k/2$ is almost identical to the proof of Dirac's theorem. Indeed, suppose that $\delta(G) \geq k/2$, and let $P = x_1, \ldots, x_\ell$ be a longest path in G. Since G does not contain a path of length k, we have $\ell \leq k \leq n-1$. As in Dirac's theorem, G cannot have an ℓ-cycle and we must have $N(x_1), N(x_\ell) \subseteq P$, and $N(x_1) \cap N^+(x_\ell) = \emptyset$, leading to a contradiction.

When $n = |V|$ is a multiple of k, this bound is best possible as the disjoint union of n/k K_ks shows.

Solution of Problem 6:

(a) Let $x_0 = (0, 0, \ldots, 0)$ be the all-zero vector of length n. Let $X := \{x \in \mathbb{F}_2^n : d(x_0, x) \text{ is odd}\}$ and $Y := \{x \in \mathbb{F}_2^n : d(x_0, x) \text{ is even}\}$. Clearly $X \cup Y = \mathbb{F}_2^n$ and $X \cap Y = \emptyset$, so X and Y partition the vertex set. We must show that there are no edges between vertices of X, and no edges between the vertices of Y. Suppose $x, x' \in X$, and write S and S' for the support of x and x', respectively. (Recall that by the support of $x \in \mathbb{F}_2^n$ we mean the subset $S \subseteq [n]$ which indexes the non-zero co-ordinates of x.) Since x and x' each contain an odd number of 1s, we know that both S and S' have odd cardinality. But if x and x' were connected by an edge, then we would have $||S| - |S'|| = 1$, which is impossible. A similar argument can be made against edges between two elements of Y.

(b) Let us first assume that $G = (V, E)$ is bipartite. Suppose that $V = X \cup Y$ is a bipartition of the vertex set. By the pigeonhole principle, any subgraph H of G must have at least half its vertices in X or at least half its vertices in Y. Suppose without loss of generality that H has at least half its vertices in X. Since G is bipartite, there are no edges between the vertices of X, hence H has an independent set containing at least half its vertices.

Conversely, suppose that G were not bipartite. By the characterisation of bipartite graphs (Theorem 5.2), this means that G must contain an odd cycle C_{2k+1} for some k. Let H be this odd cycle. By the pigeonhole principle, every set which contains at least half the vertices in H must contain at least 2 consecutive vertices of the cycle. Hence H cannot have an independent set containing at least half its vertices.

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