MATH 20002: COMBINATORICS

Solutions: Feedback Class Week 5

Solution of Problem 1:

(a) By the handshaking lemma, the number of vertices of odd degree in a graph must be even. This means there can be $0, 2, 4, 6, 8$ odd-degree vertices and $9, 7, 5, 3, 1$ even-degree vertices, respectively. Note that in all possible pairings either the number of vertices of degree 6 is at least 5, or the number of vertices of degree 5 is at least 6.

(b) We proceed by induction on v. The case $v = 1$ is trivially true. Note also that we may always assume that $e \leq v$ since otherwise the statement is trivially true (so we are only dealing with very sparse graphs here). So suppose that we have a graph G with $v \geq 2$ and that the statement holds for all graphs with $v - 1$ vertices. If there is an isolated vertex (a vertex of degree 0) in G, remove it from G to obtain a graph G' with $v - 1$ vertices and e edges. By the inductive hypothesis, this graph G' has at least $v - 1 - e$ connected components. Adding the isolated vertex back in (as its own connected component) shows that G has at least $v - 1 - e + 1 = v - e$ connected components, as required. Suppose then that every vertex in G has degree at least 1, and fix an arbitrary such vertex. Removing it (and any edges incident with it) from G yields a graph G' on $v - 1$ vertices with at most $e - 1$ edges. By the inductive hypothesis, the graph G' has at least $v - 1 - (e - 1)$ connected components, and so does G. This completes the first part of the question. In a connected graph there is only one connected component, so we need that $v - e \leq 1$ to avoid a contradiction. But this means that $e \geq v - 1$.

Solution of Problem 2: We know that isomorphisms preserve the degree sequence of a graph. This immediately shows that G_2 cannot be isomorphic to any of the other three graphs since it contains two vertices of degree 4 (and all other graphs on the list are 3-regular, i.e. $\deg(v) = 3$ for all $v \in V$). Another way to see this is to observe that the graph G_2 contains a triangle, and none of the other graphs do, so G_2 cannot be isomorphic to any of G_1, G_3 or G_4. The graph G_4 contains a cycle of length 4, while the length of the shortest cycle in G_1 and G_3 is 5, so G_4 cannot be isomorphic to G_1 or G_3. So the only possibility for an isomorphism is between G_1 and G_3. By experimentation we find the following assignment between the vertices of G_1 and those of G_3, which we can verify is indeed an isomorphism (there are numerous others):

$$1 \mapsto 6, 2 \mapsto 5, 3 \mapsto 4, 4 \mapsto 10, 5 \mapsto 7, 6 \mapsto 2, 7 \mapsto 3, 8 \mapsto 8, 9 \mapsto 9, 10 \mapsto 1.$$

Solution of Problem 3:

(a) This follows from the handshaking lemma: since $2|E| = \sum_{x \in V} \deg_G(v)$ and the left-hand side is even, the number of odd summands on the right-hand side must be even.

(b) Consider the one-sided infinite path on vertex set $\{x_n : n \in \mathbb{N}\}$ whose edges are given by $\{x_i x_{i+1} : i \in \mathbb{N}\}$. Since x_1 has degree 1 and all other vertices have even degree, the sum of the degrees $\sum_{v \in V} \deg(v)$ is odd and hence cannot equal twice the number of edges.

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