MATH 20002: COMBINATORICS

Solutions Assessed Problem Sheet 7

Solution of Problem 1:

(a) Let G be a graph on n vertices. For a vertex in G to have d neighbours, we need $d \leq n-1$ since there are $n-1$ other vertices in the graph. It follows that $d+1 \leq n$.
By the handshaking lemma, we have

$$nd = \sum_x \deg(x) = 2|E|,$$

so nd must be even.

(b) Let $A, B \subseteq V$. By $E(A, B)$ we denote the set of edges between vertices in A and vertices in B. Counting edges, we find that

$$d|S| = \sum_{v \in S} \deg(v) = 2|E(S)| + |E(S, V \setminus S)|.$$

where the first equality follows from the fact that G is d-regular and the second from analyzing the different types of edges incident with vertices in S. Similarly, we have

$$d|V \setminus S| = \sum_{v \in V \setminus S} \deg(v) = 2|E(V \setminus S)| + |E(S, V \setminus S)|.$$

But since $|S| = n/2 = |V \setminus S|$, we have $|E(S)| = |E(V \setminus S)|$.

(c) Let G be a bipartite graph on vertex classes X and Y. Suppose there is an edge in G, e say, that disconnects G. Let C be a connected component of $G \setminus e$, and denote by C_X and C_Y the vertices of C belonging to vertex classes X and Y, respectively. We may assume that $e = xy$ is such that $x \in C_X$ but $y \in Y \setminus C_Y$. Now since $G \setminus e$ is bipartite,

$$|E(C)| = \sum_{v \in C_X} \deg_{G \setminus e}(v) = d|C_X| - 1,$$

while the same reasoning gives

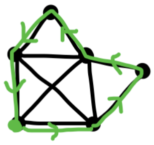
$$|E(C)| = \sum_{w \in C_Y} \deg_{G \setminus e}(w) = d|C_Y|.$$

If $d \geq 2$, then $|E(C)|$ and $|E(C)| + 1$ cannot be simultaneously divisible by d.

Solution of Problem 2:

(a) In order to show that a graph has a Eulerian circuit or a Hamiltonian cycle, you must either explicitly exhibit such a structure or quote a theorem/give an argument that proves it. If your answer is negative, then only the latter is a possibility.

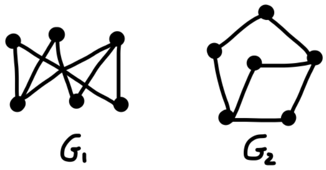
- H has a Hamiltonian cycle but no Eulerian circuit (as some of its vertices have odd degree).



An even better way to exhibit a Hamiltonian cycle is to label the vertices and write down the sequence of edges involved explicitly.

- G has neither a Hamiltonian cycle nor a Eulerian circuit since it has a vertex of degree one, from which neither a Hamiltonian cycle nor a Eulerian circuit can exit once they have reached that vertex.

(b) The following two graphs G_1 and G_2 have the desired properties (there are many other examples).



The degree sequence in each case is $2, 2, 2, 2, 3, 3$. The graph G_1 clearly has a Hamiltonian cycle. The graph G_2 has no such cycle as it consists of a C_4 and a C_5 with two edges identified, and any cycle passing through all the vertices in the graph would end up using the end vertices of those two edges more than once.

(c) This is almost identical to the argument used in the proof of Dirac's theorem (Theorem 4.20). Pick a longest path $P = x_1, x_2, \ldots, x_\ell$ in G, and suppose that $\ell \leq k$ (note that the length of this path, i.e. the number of edges it contains, is $\ell - 1 \leq k - 1$). In this case G cannot contain an ℓ-cycle: G is connected and $\ell \leq k < n$, so if it did, then we could find another vertex x outside the cycle which would have to be connected to at least one of the vertices in the cycle, which would yield a path of length $\geq k$.

Exactly as in the proof of Dirac's theorem, we may therefore assume that $x_1 x_\ell \notin E$. Moreover, we cannot have $x_1 x_i$ and $x_{i-1} x_\ell \in E$ for any $i = 2, \ldots, \ell - 1$ since otherwise we would have a cycle of length ℓ. By maximality of P we also have that $N(x_1) \subseteq \{x_2, \ldots, x_{\ell-1}\}$ and $N(x_\ell) \subseteq \{x_2, \ldots, x_{\ell-1}\}$, and in particular that $N(x_1)$ and $N(x_\ell)^+ := \{x_i : x_{i-1} \in N(x_\ell)\}$ are disjoint subsets of $\{x_2, \ldots, x_\ell\}$. But the latter is a set of size $\ell - 1 \leq k - 1$, while $N(x_1)$ and $N^+(x_\ell)$ are sets of size at least $k/2$ each. This is a contradiction, so we must have $\ell > k$ (and thus a path of length at least k).

If $k = n$, we may take a complete graph of size k, which is connected and has minimum degree $k - 1 \geq k/2$ but does not contain a path of length k.

If G is not connected, we may take the disjoint union of two complete graphs of size k (so $n = 2k > k$), which again has minimum degree $k - 1 \geq k/2$ but does not contain a path of length k.

Solution of Problem 3:

(a) Let the virtues be $V = \{v_1, v_2, \dots, v_{20}\}$. Consider a bipartite graph with vertex classes X and Y defined as follows:

$$X := \{S \subseteq V : |S| = 8, \text{ each subset } S \text{ corresponds to one student}\}$$

$$Y := \{T \subseteq V : |T| = 9, T \supseteq S \text{ for some } S \in X\}$$

We connect $S \in X$ and $T \in Y$ by an edge if and only if $T \supseteq S$. A student currently having the set of virtues S can obtain any set of virtues $T \in Y$ such that $T \supseteq S$.

The problem of imparting one additional virtue to each student such that they remain unique therefore corresponds to the problem of finding a matching from X to Y in the above graph. By the degree-constrained version of Hall's theorem (Corollary 5.7) we have a matching from X to Y if $\min_{x \in X} \deg_G(x) \geq \max_{y \in Y} \deg_G(y)$. But the degree of every $S \in X$ is 12 (since each S contains 8 virtues, so can be extended in $20 - 8 = 12$ to a set T of 9 virtues), and the degree of every $T \in Y$ is at most 9, because at most 9 students have a set of virtues $S' \subseteq T$ that is of size 8 (there are 9 ways of leaving out one element). The desired matching therefore exists.

(b) This is not difficult. Clearly if there exists a set $S \subseteq X$ such that $|N_G(S)| < |S| - d$, then since at most $|X \setminus S|$ independent edges can emanate from $X \setminus S$, we must have strictly fewer than $|S| - d + |X \setminus S| = |X| - d$ independent edges in total. Therefore a matching of deficiency d cannot exist.

If on the other hand $|N_G(S)| \geq |S| - d$ for all $S \subseteq X$, then the graph G' formed by adding a set W of d new vertices to Y and joining each one to all vertices in X must satisfy $|N_{G'}(A)| = |N_G(A) \cup W| \geq |A| - d + |W| = |A|$ for all $A \subseteq X$. It follows that G' satisfies Hall's condition, so by Hall's theorem (Theorem 5.6) we know that G' has a matching from X to $Y \cup W$. But since W is of size d, when we remove the set W we remove at most d of the $|X|$ independent edges. The remaining edges form a set of at least $|X| - d$ independent edges from X to Y.

Solution of Problem 4:

(a) We know that T has $n - 1$ edges, so by the handshaking lemma

$$\sum_x \deg(x) = 2(n-1) = 2n - 2.$$

Now suppose T has k leaves, then the remaining $n - k$ vertices have degree at least 3. This implies that

$$\sum_x \deg(x) \geq k \times 1 + (n-k) \times 3 = 3n - 2k.$$

We therefore have the inequality $2n - 2 \geq 3n - 2k$, or $k \geq (n+2)/2 = n/2 + 1$.

(b) This is by induction on $|E|$. For $|E| \leq |V| - 1$, there is nothing to prove. Let G therefore be a graph such that $|E| > |V| - 1$, and suppose that the result holds for all graphs G' on a vertex set of size $|V|$ with strictly fewer than $|E|$ edges. Note that G cannot be a tree or a forest because of the number of edges it has, so it must contain a cycle C. Let uv be any edge in C. By the inductive hypothesis, $G \setminus uv$ contains at least $(|E| - 1) - |V| + 1$ cycles, not including C (since the edge uv is missing). Adding uv back in gives at least $|E| - |V| + 1$ cycles as required.

Any comments or corrections should be sent to Dr Julia Wolf at julia.wolf@bristol.ac.uk.