MATH 20002: COMBINATORICS

Assessed Problem Sheet 7

This problem sheet is **due by 4pm on Wednesday, 21st March 2018**.

Your answers should be written up neatly and placed in the locked bin provided specifically for this purpose in the main maths building.

Please do not forget to write your name on **each page** you submit.

You are expected to attempt **all** questions, and must justify your answers carefully. Each question carries the number of marks indicated, and together they count for 10 per cent of your course mark. Discussion with your peers is allowed but answers must be written up independently.

Illegible or anonymous submissions **will not receive any credit**.

For information on late submission penalties, please see https://www.bris.ac.uk/science/undergraduates/penalties.html.

Any thesis, dissertation, essay, or other coursework must be the students own work and must not contain plagiarised material. Please refer to the information and advice at http://www.bristol.ac.uk/library/support/findinginfo/plagiarism/.

Problem 1. [15 marks] Let $d \geq 1$ be an integer. A graph $G = (V, E)$ is said to be *d-regular* if every vertex has degree d.

(a) Show that for a d-regular graph on n vertices to exist, we must have $n \geq d+1$ and nd even.

(b) Let n be even. Show that if $G = (V, E)$ is d-regular on n vertices, and $S \subseteq V$ is a set of size $n/2$, then

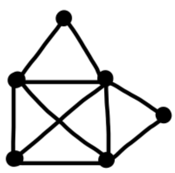
$$|E(S)| = |E(V \setminus S)|.$$

Here for $T \subseteq V$, $E(T)$ denotes the set of edges in the subgraph G' of G induced by T, that is, $G' = (T, E \cap \{xy : x, y \in T\})$.

(c) Let $d \geq 2$. Suppose that G is a d-regular connected bipartite graph. Show that one needs to remove at least 2 edges from G to make it disconnected.

Problem 2. [15 marks]

(a) For each of the graphs below, decide whether or not it has a Eulerian circuit, and whether or not it has a Hamiltonian cycle. You must justify your answers.



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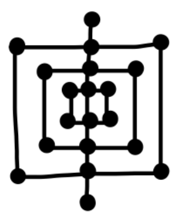

G

(b) Construct two connected graphs G_1 and G_2 with the same degree sequence such that G_1 has a Hamiltonian cycle and G_2 does not have a Hamiltonian cycle.

(c) Let G be a graph of order $n \geq 3$. Suppose that $\delta(G) \geq k/2$ for some $k < n$, and that G is connected. Show that G contains a path of length k.
Give examples to show that this statement can be false for $k = n$, or if G is not connected.

Problem 3. [12 marks]

(a) Scholars have identified twenty fundamental human virtues, including honesty, generosity, diligence, modesty, kindness and completing assessed homework assignments on time. At the start of Teaching Block 2, every student taking Combinatorics possessed exactly eight of these virtues. Furthermore, every student was unique in the sense that no two students possessed exactly the same set of virtues. The instructor must select one additional virtue to impart to each student by the end of the semester. Prove that there is a way to select an additional virtue for each student so that every student is still unique by the end of the semester.
Hint: Define a suitable bipartite graph and apply Hall's theorem.

(b) Let $d \geq 0$ be an integer, and let G be a bipartite graph with vertex classes X and Y. A set F of edges from X to Y is called *independent* if no two distinct edges from F have a vertex in common. A *matching of deficiency d from X to Y* is a set of at least $|X| - d$ independent edges between X and Y. Prove that G has a matching of deficiency d from X to Y if and only if $|N(S)| \geq |S| - d$ for all $S \subseteq X$.
Hint: You may wish to form a new graph G' by adding d new vertices to Y, and joining each one to all vertices in X.

Problem 4. [8 marks]

(a) Let T be a tree on n vertices such that no vertex has degree 2. Show that T has strictly more than $\lceil n/2 \rceil$ leaves.

(b) Show that a graph $G = (V, E)$ contains at least $|E| - |V| + 1$ cycles.

Any questions or corrections should be sent to Dr Julia Wolf at julia.wolf@bristol.ac.uk.