MATH 592 SPRING 2010
TOPICS IN ERGODIC THEORY

PROBLEM SET 3

1. As is to be expected, Bohr sets are extremely non-uniform. Show that if $\rho \leq 1/6$, then for all $t \in S$ we have

$$|\widehat{1_{B(S,\rho)}}(t)| \geq \beta/2,$$

where β denotes the density of $B(S,\rho)$. Show also that $\beta \leq 4/|S|$.

2. Prove the Vitali covering lemma. That is, show that any collection $\mathcal{I}$ of intervals contains a finite subcollection $I_1, I_2, \ldots, I_n$ of disjoint intervals such that

$$\text{meas}\left(\bigcup_{i=1}^{n} I_i\right) \geq \frac{1}{5}\text{meas}\left(\bigcup_{I\in\mathcal{I}} I\right).$$

Can you improve the constant $1/5$ to $1/2$?

3. Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$ containing 0. Show that there exists a set S of frequencies that satisfies $A \cap B(S, 1/4) = \{0\}$ and $|S| \leq 1 + \log_2 |A|$.

4. Deduce Varnavides's theorem from Roth's theorem, i.e. show that for any subset A of density α we have

$$\mathbb{E}_{x,d} 1_A(x) 1_A(x+d) 1_A(x+2d) \geq c(\alpha).$$

[You can try this problem in any of the settings $[N]$, $\mathbb{Z}/N\mathbb{Z}$ or $\mathbb{F}_p^n$.]

5. Let A be a set of n distinct squares. Show that $|A+A|/|A|$ goes to infinity with n. [Hint: A famous result of Euler states that there are no four squares in arithmetic progression.]

6. Show that one can colour $\mathbb{Z}/N\mathbb{Z}$ by $\exp(c\sqrt{\log N})$ many colours such that none of the colour classes contain a proper arithmetic progression of length 3.

7. Show that $r_3(\mathbb{F}_3^n) \geq 2^n$. Do a literature search to find out if a better result is known.

Please send any comments or corrections to jwolf137@math.rutgers.edu.