MATH 592 SPRING 2010
TOPICS IN ERGODIC THEORY

PROBLEM SET 2

1. Let A, B, C and D be subsets of an Abelian group G, each of size N. Suppose that there are ηN^3 quadruples a, b, c, d with $a \in A, b \in B, c \in C$ and $d \in D$ such that $a + b = c + d$. Show that A contains at least $\eta^4 N^3$ solutions to $a_1 + a_2 = a_3 + a_4$.

2. Suppose that $A \subseteq \mathbb{Z}$ has size N and contains at least δN^2 arithmetic progressions of length 3. Show that A has large additive energy, i.e. that it contains at least $\delta^2 N^3$ additive quadruples.

3. Let $A \subseteq \mathbb{Z}_N$ have density α. Suppose A contains $(\alpha^4 + c)N^3$ additive quadruples. Without using Fourier analysis, show that A contains an arithmetic progression of length 3, provided that c is sufficiently small.

4. Show that the U^k norms are nested, i.e. that for any integer $k \geq 2$, we have

$$\|f\|_{U^k} \leq \|f\|_{U^{k+1}}.$$

5. Define the Gowers inner product $\langle f_\epsilon \rangle_{\epsilon \in \{0,1\}^k}$ of 2^k functions $(f_\epsilon)_{\epsilon \in \{0,1\}^k}$ by appropriately modifying the definition of the U^k norm. Prove the Gowers-Cauchy-Schwarz inequality

$$|\langle f_\epsilon \rangle_{\epsilon \in \{0,1\}^k}| \leq \prod_{\epsilon \in \{0,1\}^k} \|f_\epsilon\|_{U^k}.$$

Use it to prove that the U^k norm indeed satisfies the triangle inequality.

6. Formally state and prove an inverse theorem for the U^2 norm over $\mathbb{F}_p^n$.

7. Suppose that $f : \mathbb{F}_p^n \to [-1, 1]$ is such that $|\mathbb{E}_x f(x) \omega^{x^T M x}| \geq \delta$ for some symmetric matrix M with entries in $\mathbb{F}_p$. Show that $\|f\|_{U^3} \geq \delta$.

Please send any comments or corrections to jwolf137@math.rutgers.edu.