Chromatic number, clique subdivisions, and the conjectures of Hajós and Erdős-Fajtlowicz

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Hajós conjecture

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Erdős and Fajtlowicz in 1981 showed that:

almost all graphs are counterexamples!

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The random graph $G = G(n, 1/2)$ almost surely satisfies

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They further conjectured that

the random graph is essentially the strongest counterexample!

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Theorem: (F.-Lee-Sudakov)

The Erdős-Fajtlowicz conjecture is true.

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Theorem: (F.-Lee-Sudakov)

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2. If $\alpha = a\log n$ for some $a \geq 2$, then $f(n,\alpha) \geq c\sqrt{\frac{n}{a\log a}}$.

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$$f(n, \log n) = \Theta(n^{1/2}).$$

Tight for $G(n, 1/2)$.

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If $G = (V; E)$ has $|V| = n$ and edge density d with $d^2n \geq 1600$, then there is $U \subset V$ with $|U| \geq dn/50$ such that every pair of vertices in U has at least $10^{-9}d^5n$ internally vertex-disjoint paths of length 4 which uses only vertices from $V \setminus U$ as internal vertices.

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Lemma 3

If $s \leq \rho^{\alpha-1} n$, then every n-vertex G with $\alpha(G) = \alpha$ has a vertex subset S of order s with $\leq \rho s^2$ nonadjacent pairs of vertices.

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Lemma 4

Let I be a maximum independent set of G with $|I| = \alpha$, and $U \subset V \setminus I$ such that each vertex in U has at most $d|I|$ neighbors in I. Then there is $W \subset U$ with $|W| \geq \left(\frac{e}{d}\right)^{-d\alpha} |U|$ such that every independent subset of W has order at most $d\alpha$.

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