On Graphs, Integers and Communication

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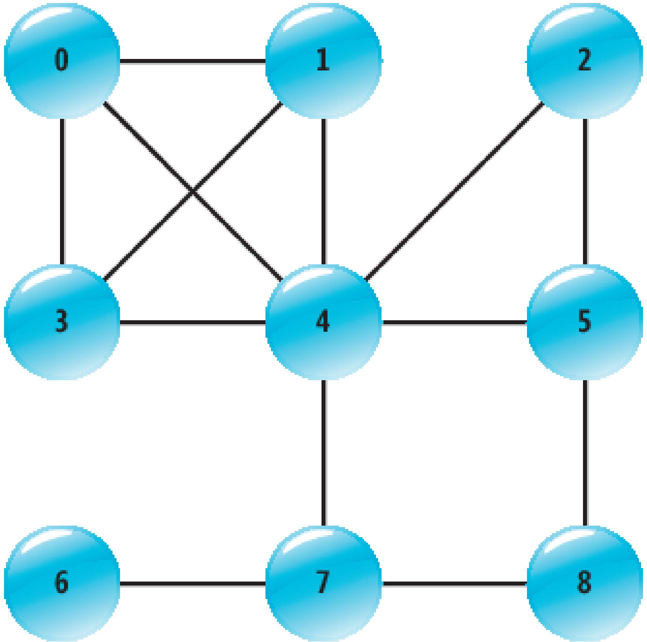

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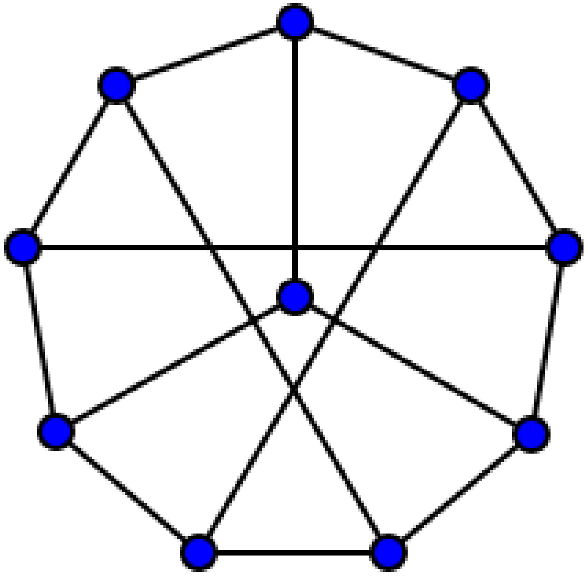
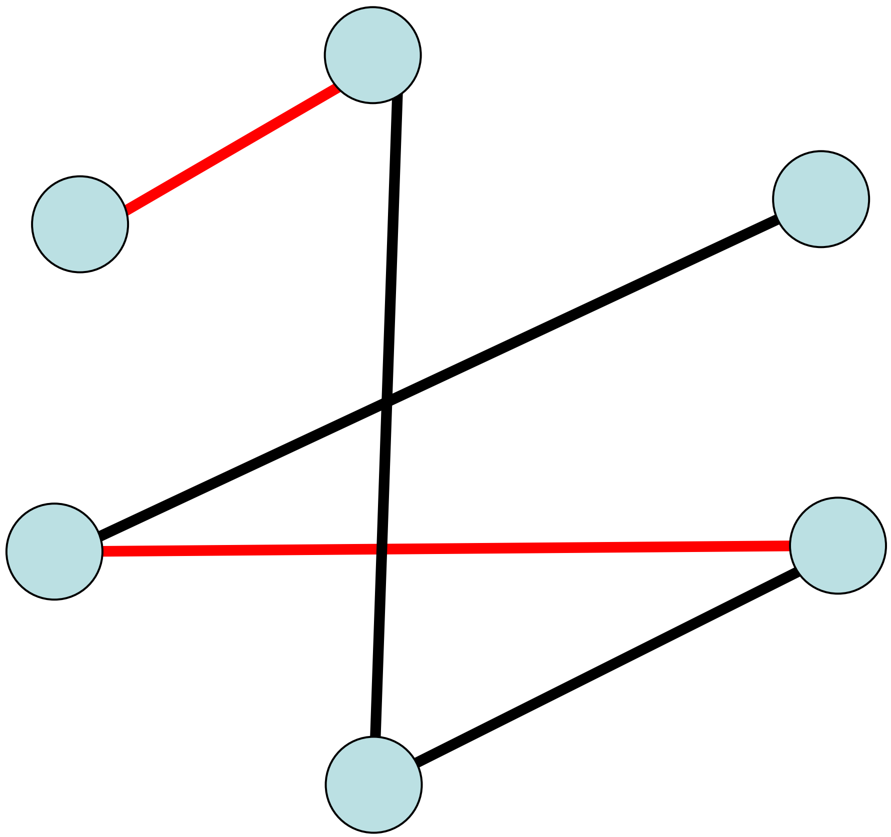
I. Induced matchings

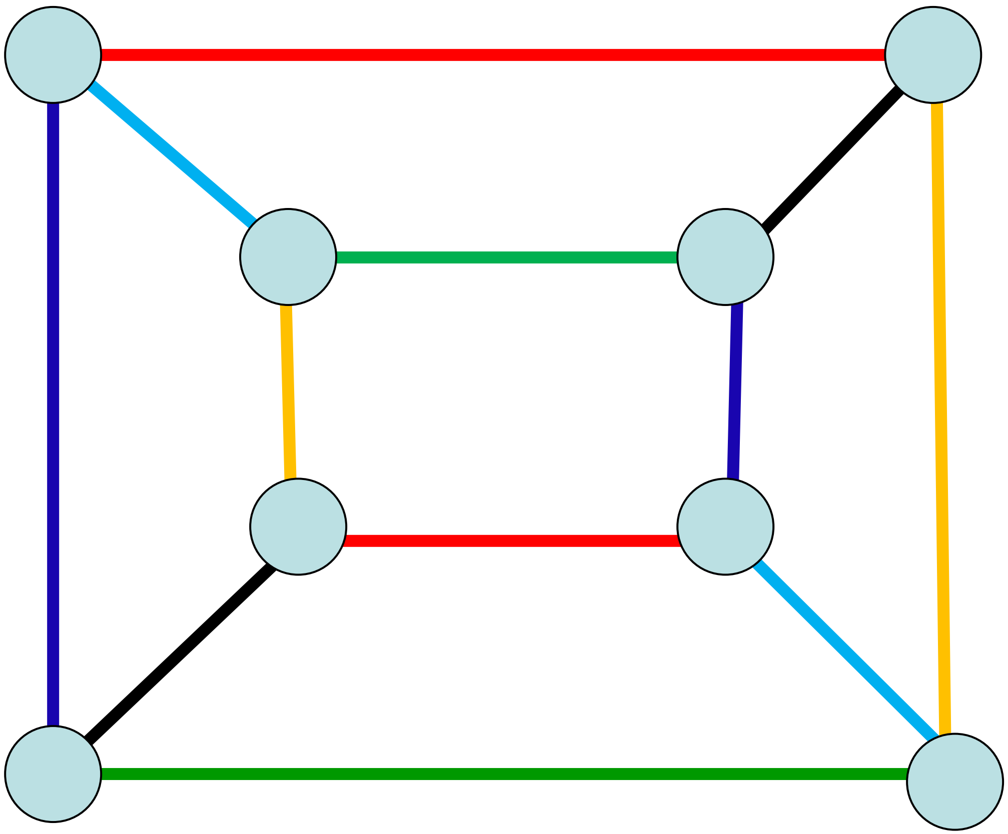
A graph G is an ordered pair (V,E), where V is a finite set of vertices, and E is a set of pairs of elements of V, called edges.



An induced matching in a graph G is a set of isolated edges, with no other edge of G having both ends in their endpoints



Question: How **dense** can a graph be, if its set of edges is a disjoint union of **large** pairwise disjoint induced matchings ?



Ruzsa+Szemerédi (78): If G on n vertices is the edge disjoint union of induced matchings, each of size at least cn, then the number of these matchings is at most o(n)

Here $o(n) \le O(n/(\log^* n)^{1/5})$

Fox(11): In fact $o(n) \le O(n/\log\log\log \ldots \log n)$, [$O(\log(1/c))$ times iterated logarithm].

II. 3-term arithmetic progresions

The Ruzsa-Szemerédi result implies a theorem of Roth (54) in additive number theory: If X is a subset of Z_n containing no 3-term arithmetic progression, then |X|=o(n).
(Same holds for any abelian group of odd order).

Proof: Given a subset X of Z_n with no 3-term AP, construct a bipartite graph on the sets of vertices $A=Z_n$ and $B=Z_n$ consisting of n matchings M_a, $a \in Z_n$

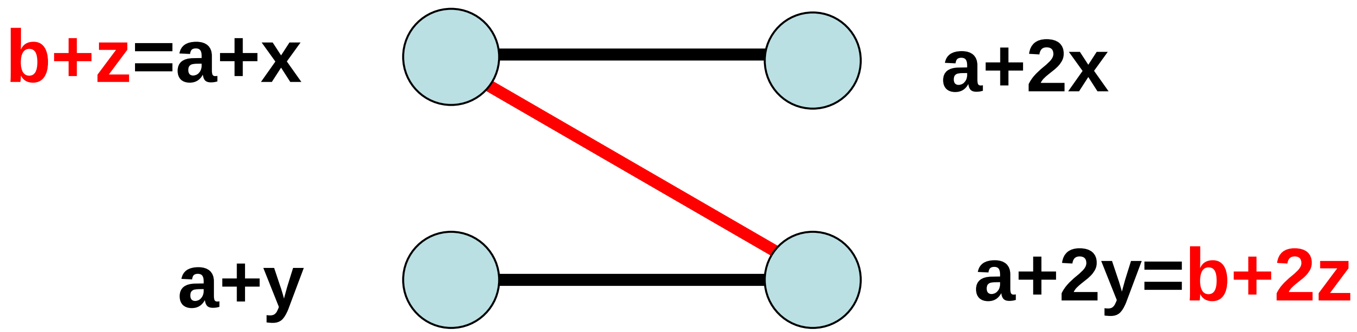
For every a in Z_n , M_a is the matching of all edges (a+x,a+2x) with x in X. Thus $|M_a|=|X|$.

Fact 1: The matchings M_a (a in Z_n) are pairwise disjoint.

Indeed, a+x and a+2x determine a and x.

Fact 2: If X contains no 3-term AP, then each M_a is an induced matching.

Indeed, otherwise:



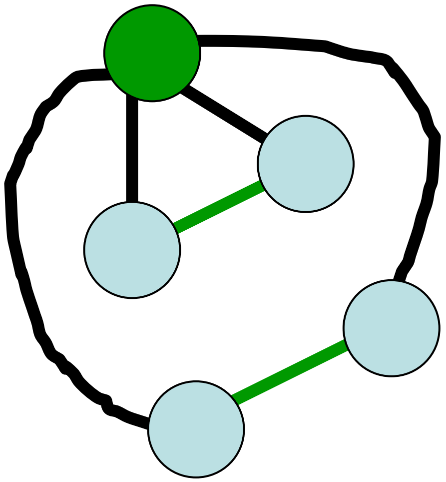
Implying that z=2y-x, that is, z+x=2y and x,y,z is a 3-term AP.

Thus, the graph has 2n vertices and n induced matchings, each of size |X|, implying that |X|=o(n).■

Behrend (46): **There is a subset X of Z_n with no 3-term AP, of size $|X| \geq n / e^{O(\sqrt{\log n})}$ $=n^{1-o(1)}$.**

Therefore, there is a graph with n vertices and $n^{2-o(1)}$ edges consisting of Ω(n) pairwise edge disjoint matchings, each of size $n^{1-o(1)}$.

By connecting the endpoints of each induced matching M_a to a new point a', we get a graph with O(n) vertices and $n^{2-o(1)}$ edges, in which every edge lies in a unique triangle.

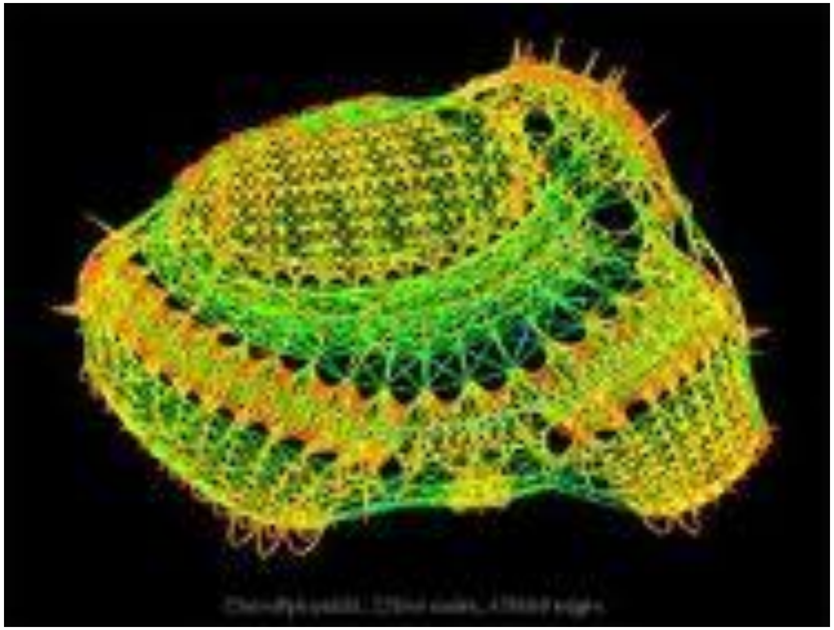


This gives a graph with n vertices and $3\epsilon n^2$ edges, from which one has to delete ϵn^2 edges to destroy all triangles, and the graph contains only

$$\epsilon^{(\log(1/\epsilon))} n^3$$

triangles.

III. Graph Property Testing



Objective [Goldreich, Goldwasser, Ron (96)]: **distinguish between graphs on n vertices that satisfy a property P, and ones that are ε-far from satisfying it, by inspecting the induced subgraph on a random sample of only f(ε) vertices.**

Here ε-far means that one has to delete or add to the graph at least εn^2 edges to get a graph satisfying the property.

A graph property is called testable if a random sample of f(ε) vertices suffices. It is easily testable if f(ε) is polynomial in 1/ε.

It is known that for any fixed graph H, the property of being H-free (= containing no copy of H) is testable. [ADLRY (92), AFKS (00), AS (05)]

By the consequence of the Behrend construction, if H is a triangle, this property is not easily testable.

In fact:

Thm (A): For a fixed graph H, the property of being H-free is easily testable if and only if H is bipartite.

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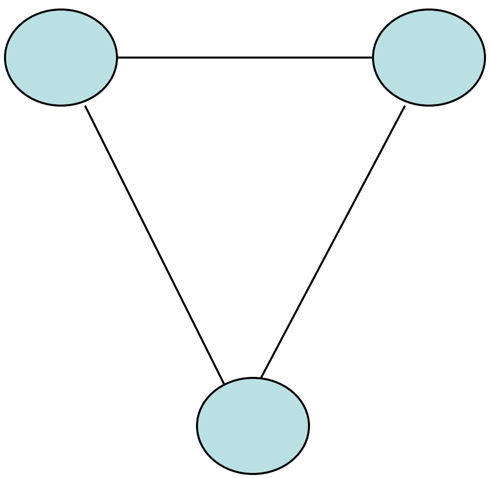
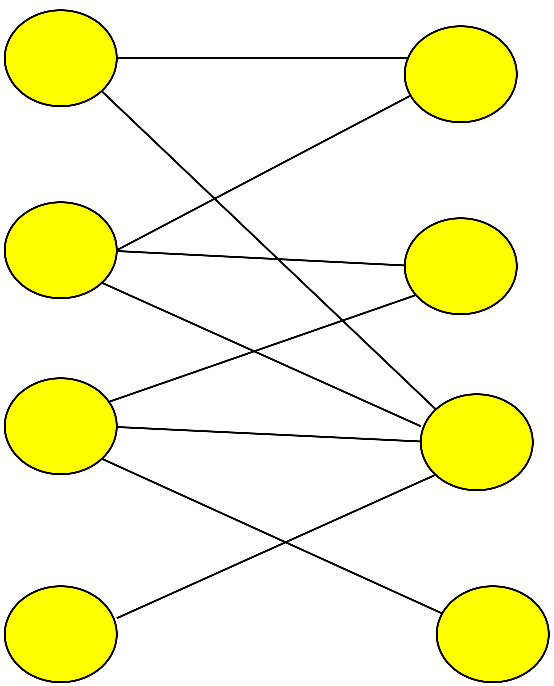
H_1 : not easy



H_2 : easy



Open: characterize all the easily testable graph properties

GGR(96): Being k-colorable is easily testable [Goldreich-Trevisan: so are some more general partition properties]

A-Shapira(06), A-Fox(12): Being induced H-free is easily testable iff H is a path of length 1,2,3 or its complement (with one possible exception).

A+Fox (12): Being a perfect graph and being a comparability graph are not easily testable.

IV. Nearly complete graphs

Recall:

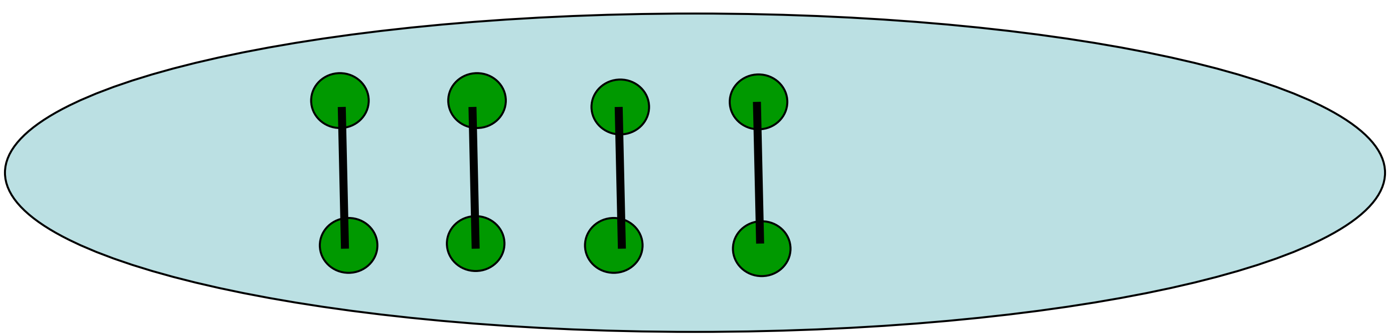
Question: How dense can a graph be, if its set of edges is a disjoint union of large pairwise disjoint induced matchings ?

RS+Behrend: If the graph has n vertices, and each matching is of size at least $\Omega(n)$, then the number of matchings is $o(n)$, but if each matching is of size $n^{1-o(1)}$ then there may be $\Omega(n)$ matchings.

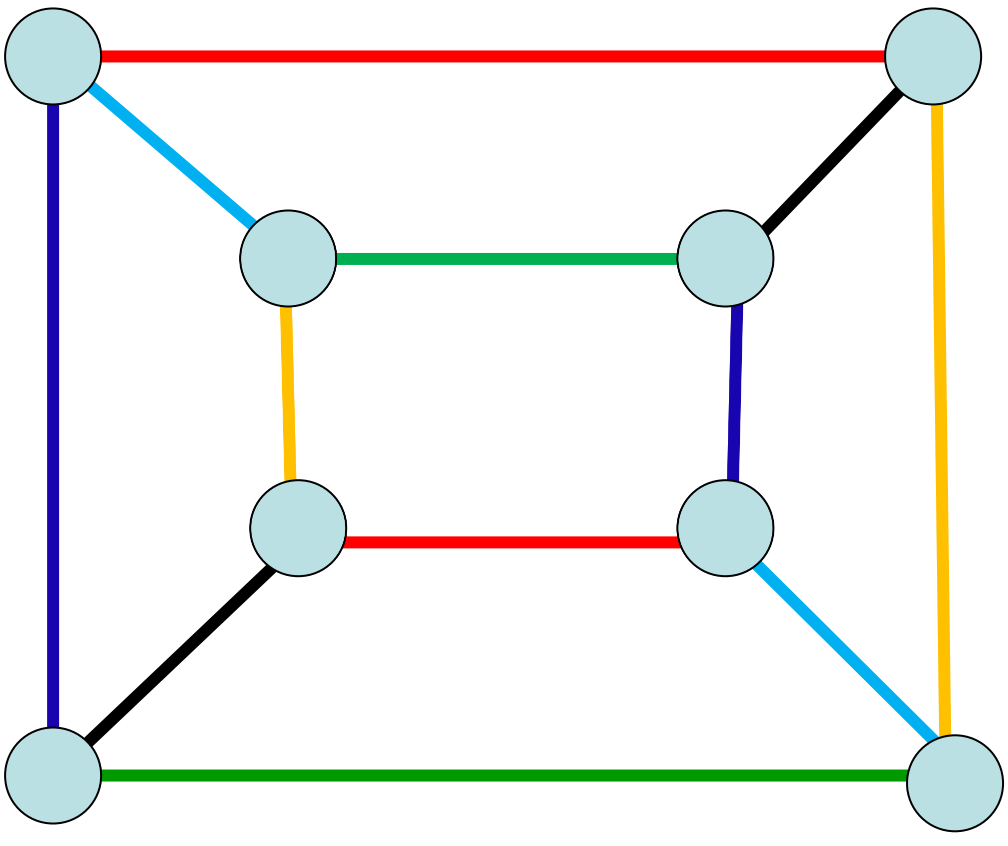
Trivial: If the graph has n vertices and each matching contains >n/3 edges, then the number of induced matchings cannot exceed 1.

Easy: If each matching contains >(1/4+ε) n edges, then the number of matchings is at most C(ε). [FNRRS(02): can have much more if each matching contains (1/4-ε) n edges]

In general, it seems that if the matchings are large, the graph cannot be dense.



Conjecture (Meshulam): no graph on n vertices and $\Omega(n^2)$ edges can be the edge disjoint union of induced matchings, each of size at least n^{ε}.



Thm 1 [A, Moitra+Sudakov (12)]:

There exists a graph on n vertices with

$$(1-o(1))\binom{n}{2}$$

edges, which is the edge disjoint union of induced matchings, each of size at least

$n^{1-o(1)}$.

Thm 2 [A, Moitra+Sudakov (12)]:

For any $\varepsilon>0$, one can cover the complete graph on n vertices by $C(\varepsilon)$ graphs G_i , so that the edges of each G_i can be covered by at most $n^{1+\varepsilon}$ induced matchings .

The proofs combine geometric tools, based on measure concentration in high dimensional Euclidean spaces, with probabilistic methods, and techniques from the theory of error correcting codes.

The first construction is based on ideas of Fox and Loh (11), following Behrend (46).

V. Shared communication channels

A shared channel consists of n sources s_i and n destinations t_j .

Source s_i has to transmit message m_{ij} to destinations t_j , $(1 \leq i, j \leq n)$.

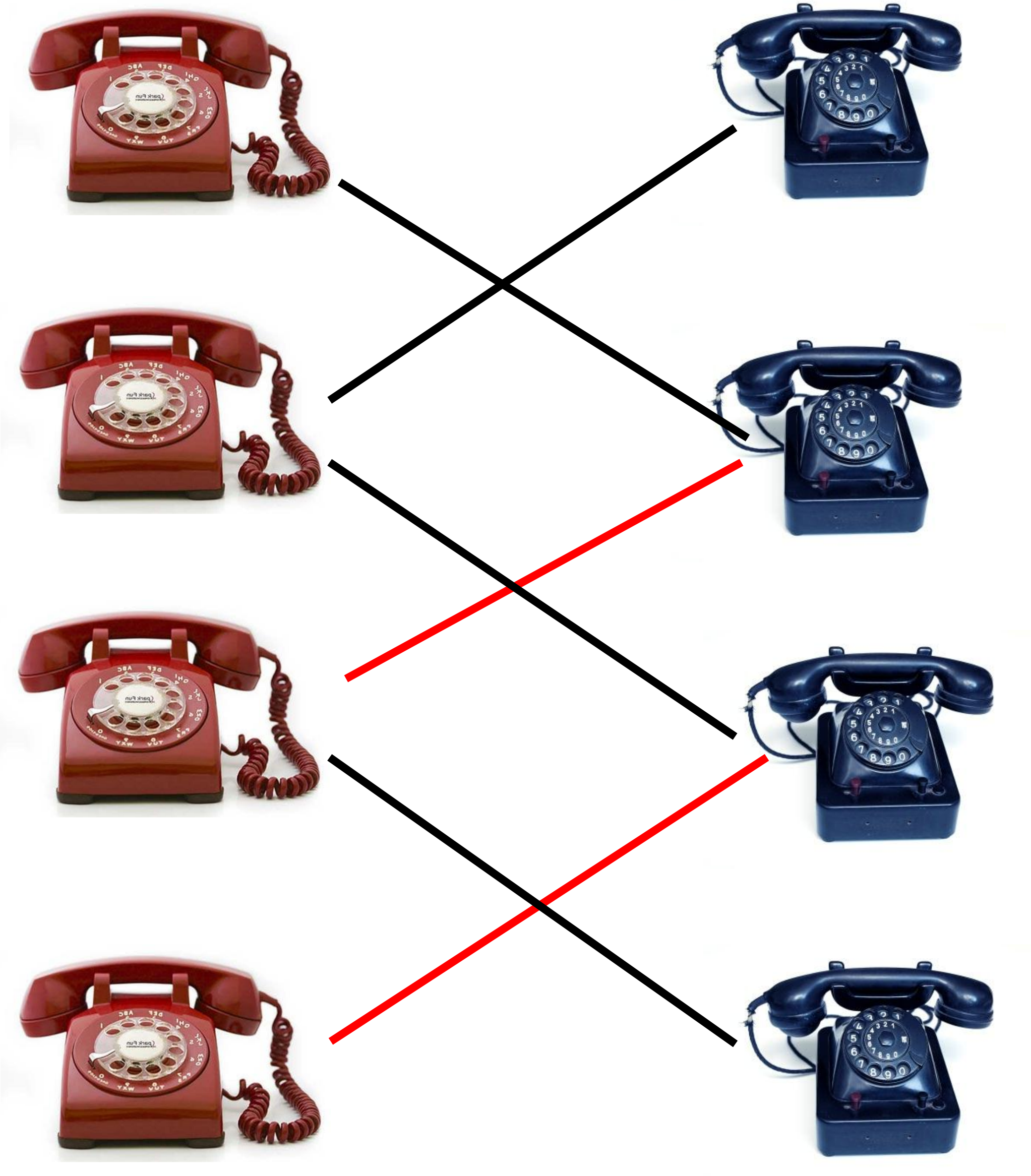
This is implemented by constructing several interconnection networks G_p between the sources and the destinations.

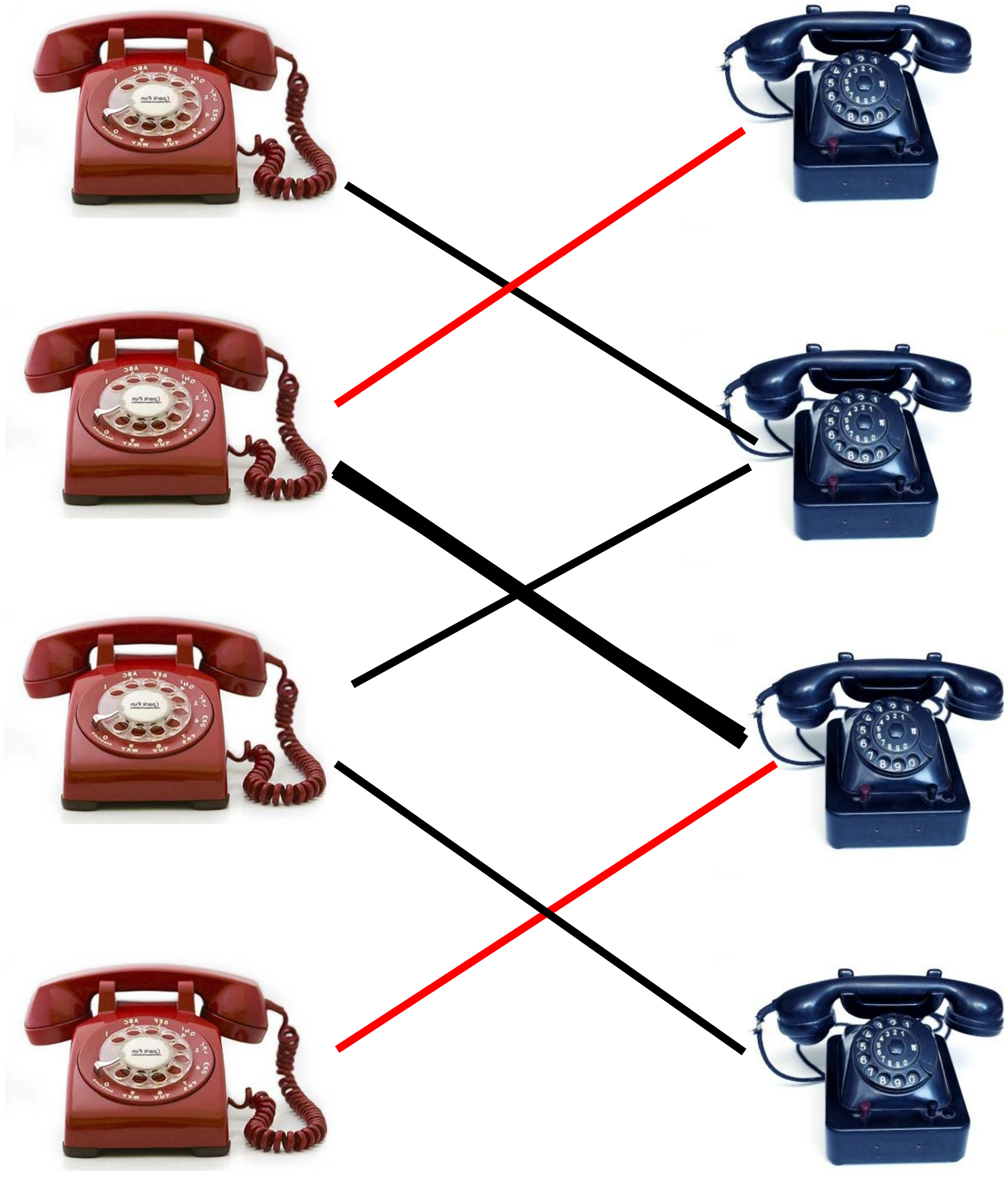
Each G_p is a bipartite graph with vertex classes S and T, where S is the set of sources and T is the set of destinations.

A message transmitted in G_p by s_i reaches all destinations t_j connected to it.

A message m_{ij} is received successfully by t_j in G_p in a given round if it is being sent by s_i and no other source connected to t_j transmits in this round.

Therefore, the set of messages m_{ij} received successfully in a round forms an induced matching in G_p .





Therefore, the objective is to cover the complete bipartite graph with n vertices in each class by a small number r of graphs G_p so that each G_p can be covered by a small number of induced matchings.

Clearly:

(i) For every r, the total number of required induced matchings is at most n^2 .

(ii) For every r, the total number of induced matchings is at least n.
(this lower bound can be slightly improved)

Birk, Linial + Meshulam (93): for r graphs, the total number of required induced matchings does not exceed $O(n^2 / (\log n)^{r-1})$.

A, Moitra+Sudakov (12):

(i) For r graphs, the total number of required induced matchings does not exceed $n^{1+O(1/\log r)}$.

(ii) For r=2 graphs, the total number of required induced matchings does not exceed $n^{1.94}$ (and is at least $n^{4/3}$.)

VI. Open:

Improved bounds: for each given n and t, estimate the maximum possible density of a graph on n vertices which is the edge disjoint union of induced matchings, each of size t.

This may supply improved bounds for the max. density of a subset of {1,2,…,n} with no 3-term arithmetic progression, may provide better shared communication channels and may yield new results in graph property testing.