The minimum number of monochromatic 4-term progressions

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May 28, 2009

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Note that $\widehat{1_A}(0) = \alpha$ whenever $A \subseteq \mathbb{Z}_p$ of density α.

Counting monochromatic 3-term progressions

Fact

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If $\mathbb{Z}_p$ is 2-colored, then there are at least $\frac{1}{8}p^2$ monochromatic 3-term progressions.

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since $\widehat{1_A}(t) = -\widehat{1_{A^C}}(t)$ for $t \neq 0$.

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- The Fourier transform is not sufficient for counting 4-term progressions in dense sets. $\rightarrow$ quadratic Fourier analysis
- Because we are using 2 colors only, the coloring problem is closely related to density problems such as Szemerédi's theorem for longer progressions.

Counting monochromatic 4-term progressions

Theorem

- *There exists a 2-coloring of* $\mathbb{Z}_p$ *with fewer than*

$$\frac{1}{16}\left(1-\frac{1}{2025}\right)p^2$$

monochromatic 4-term progressions.

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- *Any 2-coloring of* $\mathbb{Z}_p$ *contains at least*

$$\frac{1}{32}p^2$$

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Counting 3-term arithmetic progressions in dense sets

Definition

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$$A = \{x \in \mathbb{Z}_p : |x^2| \quad \text{small}\}$$

$$x^2 - 3(x+d)^2 + 3(x+2d)^2 - (x+3d)^2 = 0$$

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Question

Are there any subsets of $\mathbb{Z}_p$ that are uniform but contain FEWER than the expected number of 4-term progressions?

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- On the interval $[1, 18]$, let f take successive values

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so that

$$\sum_{x,d} f(x)f(x+d)f(x+2d)f(x+3d) = -36.$$

Gowers's construction

- Define a function $F : \mathbb{Z}_p \to \{-1, 0, 1\}$ by setting $F(x) = f(t)$ whenever $x \in I_t$, where I_t stands for the interval $[(2t-1)m, 2tm]$ and m is a positive integer between $p/(5 \times 18)$ and $p/(4 \times 18)$.

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- The 4-AP counts of F and f are related via

$$\sum_{x,d} F(x)F(x+d)F(x+2d)F(x+3d)$$

$$= s \sum_{x,d} f(x)f(x+d)f(x+2d)f(x+3d),$$

where $s \geq m^2/9$.

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With high probability the resulting set A is uniform but contains at most

$$1/16(1 - 36/9(5 \times 18)^2)p^2$$

4-term progressions.

What does this mean for monochromatic progressions?

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with the number of 4-APs $p_4(A)$ in A being less than the expected $\alpha^4/2$.

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Lemma

With the c_i defined as above, we have that

$$4(c_0 + c_4) + (c_1 + c_3) = 4(1 - 3\alpha + 3\alpha^2)$$

for any coloring of $\mathbb{Z}_p$ in which the red color class has density α.

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The preceding lemma together with the identity $\sum_{i=0}^{4} c_i = 1$ now implies that

$$(*) \qquad c_0 + c_4 = \frac{1}{3}c_2 + (1 - 4\alpha + 4\alpha^2).$$

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This together with $(*)$ yields a lower bound on $c_0 + c_4$.

The crucial observation

- Any 3-term progression S of the form $x, x+d, x+2d$ determines a unique (unordered) pair of points (a, b) such that the five points and each of the quadruples $a, x, x+d, x+2d$ and $x, x+d, x+2d, b$ lie in arithmetic progression.

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- Note that in these statements we have used the assumption that p is prime.
- Two 4-APs containing S have different color parities if and only if the frame pair of S is bichromatic.
- The total number of monochromatic pairs is at its minimum for densities close to $1/2$.

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Question

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If the edges of K_n are 2-colored, what is the minimum number of monochromatic K_4s?

- It was conjectured by Erdős that the number of monochromatic K_4s is always at least the number expected in the random case.
- It is not difficult to see that this is true for triangles.

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- The initial construction is rather obscure, using a quadratic form over a finite field.
- It was subsequently simplified and rephrased.
- Now many examples are known computationally, but a theoretical framework is lacking.

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Given a 2-coloring of K_n the minimum number of monochromatic K_4s is at least

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The proof proceeds via ingenious combinatorial counting and an optimization.

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- A wide gap between upper and lower bounds persists for the problem in both sets and graphs.
- It would be interesting to make the analogy between the two more precise.
- For the integers $1, 2, \ldots, n$ instead of $\mathbb{Z}_p$, the problem is unsolved even for 3-term progressions. $\rightarrow$ Schur triples

Bibliography

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